Blue Print (As per PU Board)

Topic	1 mark questions	2 marks questions	3 marks questions	5 marks questions	Total Marks
Application of Derivatives	–	1	1	1	10

Two marks questions

1. Using differentiation find the approximate value of $\sqrt{49.5}$

Answer: Let $y=\sqrt{x}, x=49$ and $\Delta x=0.5$

$\therefore \Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{49.5}-\sqrt{49}=\sqrt{49.5}-7$ (1 mark)

$\therefore \sqrt{49.5}=\Delta y+7$, dy is approximately equal to Δy

$$dy=\frac{dy}{dx}\Delta x=\frac{1}{2\sqrt{x}}(0.5)=\frac{1}{2\sqrt{49}}\times 0.5=\frac{0.5}{14}=0.035$$

$\therefore$ approximate value of $\sqrt{49.5}=0.035+7=7.035$ (1 mark)

2. Find the slope of the tangent to $y=\frac{x-1}{x-2}, x\neq 2$ **at** $x=10$

Answer: $y=\frac{x-1}{x-2}$

$$\frac{dy}{dx}=\frac{(x-2)(1)-(x-1)(1)}{(x-2)^2}=\frac{-1}{(x-2)^2}$$ (1 mark)

Slope of tangent at $x=10$, $\frac{dy}{dx}_{(x=10)}=\frac{-1}{(10-2)^2}=\frac{-1}{8^2}=\frac{-1}{64}$ (1 mark)

3. Find the local maximum value of $g(x)=x^3-3x$

Answer: $g(x)=x^3-3x$

Differentiating, $g'(x)=3x^2-3$

Differentiating, $g''(x)=6x$ (1 mark)

$g'(x)=0\Rightarrow 3x^2-3=0\Rightarrow x=\pm 1$

$x=-1, g''(x)=-6<0$

$\therefore$ local maximum exist at $x=-1$

$\therefore$ Local maximum $\Rightarrow g(-1)=(-1)^3-3(-1)=2$ (1 mark)

4. Find a point on the curve $y=x^3-11x+5$ **at which the tangent is** $y=x-11$

Answer: $y=x-11\Rightarrow$ slope of tangent $=1=\frac{dy}{dx}$ (1 mark)

$y=x^3-11x+5$

$\frac{dy}{dx}=3x^2-11$ $\therefore 3x^2-11=1\Rightarrow x^2=\frac{12}{3}=4$ $\therefore x=\pm 2$

When $x=2, y=2^3-11\times 2+5$

$x=-2, y=(-2)^3-11(-2)+5=-19$

$\therefore$ The points is $(2,-9)$ and point $(-2,19)$ does not lie on $y=x-11$ (1 mark)

Three marks questions

5. Find 2 positive numbers whose sum is 15 and the sum of whose square is minimum

Answer: Let x and y be 2 nos

$\therefore x+y=15 \qquad \Rightarrow y=15-x$ (1 mark)

$\therefore x^2+y^2=S$ is minimum (given)

$s=x^2+(15-x)^2=2x^2-30x+225$

$\frac{ds}{dx}=4x-30,\ \frac{d^2s}{dx^2}=4$

$\frac{ds}{dx}=0\Rightarrow 4x=30\ \therefore x=\frac{15}{2}$

$\left(\frac{d^2s}{dx^2}\right)_{(x=15/2)}=4>0$ (1 mark)

$\therefore$ minimum value occurs at $x=\frac{15}{2}$

$\therefore y=15-x=15-\frac{15}{2}=\frac{15}{2}$

2 nos are $\frac{15}{2},\frac{15}{2}$ (1 mark)

6. **Find** 2 **positive numbers** x and y **such that** $x+y=60$ **and** xy^3 **is maximum**

Answer: $x+y=60\Rightarrow x=60-y$

Let $S=xy^3 \quad S=(60-y)y^3=60y^3-y^4$

$\left.\begin{array}{l}\frac{ds}{dy}=180y^2-4y^3\\ \frac{d^2s}{dy^2}=360y-12y^2\end{array}\right]$ (1 mark)

$\frac{ds}{dy}=0\Rightarrow 180y^2-4y^3=0\Rightarrow 4y=180\therefore y=45$ (1 mark)

$\frac{d^2s}{dy^2}=360\times 45-12(45)^2=16200-24300=-8100$

$\frac{d^2s}{dy^2}<0.$ S has local maximum

$x=60-y$

$x=60-45=15$

$\therefore$2 nos are 15 and 45 (1 mark)

7. Find the local maximum value of $\sin x+\cos x,\ 0<x<\frac{\pi}{2}$

Answer: $f(x)=\sin x+\cos x, x\in R$

$f'(x)=\cos x-\sin x$

$f''(x)=-\sin x-\cos x$ (1 mark)

Common condition for maximum or minimum is $f'(x)=0$

$\therefore \cos x-\sin x=0\Rightarrow \tan x=1\ \therefore x=\frac{\pi}{4}$ (1 mark)

$f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}-\cos\frac{\pi}{4}=\frac{-1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\frac{-2}{\sqrt{2}}=-\sqrt{2}<0$

$f(x)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$

$\therefore$ the maximum value is $\sqrt{2}$ (1 mark)

8. **Prove that the curves $x=y^2, xy=k$ cut at right angles if $8k^2=1$**

Answer: $x=y^2$

Differentiating $1=2y\frac{dy}{dx}$ $\therefore \frac{dy}{dx}=\frac{1}{2y}$ $\ldots(1)$

$xy=k$

Differentiating, $x\frac{dy}{dx}+y=0$ $\Rightarrow \frac{dy}{dx}=\frac{-y}{x}$ $\ldots(2)$ (1 mark)

For orthogonal curves, $(1)\times(2)=-1$

$\therefore\left(\frac{1}{2y}\right)\left(\frac{-y}{x}\right)=-1$

$\Rightarrow x=\frac{1}{2}$ (1 mark)

$\therefore x=y^2 \Rightarrow y^2=\frac{1}{2}$

$xy=k\Rightarrow x^2y^2=k^2$

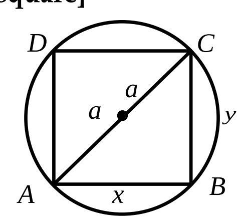
$\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)=k^2$

$\Rightarrow 8k^2=1$ (1 mark)

Five marks questions

9. **Show that of all rectangles inscribed in a given fixed circle the square has the maximum area. [show that the maximum rectangle inscribed in a circle is always a square]**

Answer: Let ABCD be a rectangle with sides $x\,\&\,y$ which is inscribed in a circle of radius 2a



$\therefore x^2+y^2=4a^2 \Rightarrow y^2=4a^2-x^2$

Let A be the area of rectangle

$\therefore A=xy\Rightarrow A^2=x^2y^2$ $\quad Let\ A^2=S$

(1 mark)

$S=x^2y^2$

$S=x^2\left(4a^2-x^2\right)=4a^2x^2-x^4$ $\left(\because y^2=4a^2-x^2\right)$

$\frac{ds}{dx}=8a^2x-4x^3$

$\frac{d^2s}{dx^2}=8a^2-12x^2$ (1 mark)

Common condition for maximum or minimum is $\frac{ds}{dx}=0$

$\therefore 4x\left(2a^2-x^2\right)=0\Rightarrow x=\sqrt{2}\,a$ (1 mark)

At $x=\sqrt{2}\,a,\ \frac{d^2s}{dx^2}=8a^2-24a^2=-16a^2<0$

$\therefore$ we have maximum at $x=\sqrt{2}a$
Area is maximum at $x=\sqrt{2}a$ (1 mark)
$y^2=4a^2-x^2=4a^2-2a^2=2a^2$

$\therefore y=\sqrt{2}a$
$\therefore x=y=\sqrt{2}a$
Hence square has maximum area as a rectangle inscribed in a circle. (1 mark)

10. The length x of a rectangle is decreasing at the rate of 3 cm/min and the width y is increasing at the rate of $2cms/\min$. When $x=10cm$ and $y=6cm$, find the rate of
(i) the perimeter (ii) the area of the rectangle.

Answer: Since the length x is decreasing, $\frac{dx}{dt}=-3cms/\min$

Width y is increasing, $\frac{dy}{dt}=2cm/\min$ (1 mark)

(i) Perimeter, $P=2(x+y)$

Differentiating wrt 't' $\frac{dp}{dt}=2\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$ (1 mark)

$=2(-3+2)=-2$

$\therefore$ Perimeter is decreasing at the rate of 2 cm/min (1 mark)

(ii) Area of rectangle, $A=xy$

Differentiating wrt 't', $\frac{dA}{dt}=x\frac{dy}{dt}+y\frac{dx}{dt}$ (1 mark)

$=10(2)+6(-3)=2cm^2/\min$ (1 mark)

Area is increasing at the rate of $2cm^2/\min$

11. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of an edge is 10 cms

Answer: Let V be the volume and S be the surface area of a cube with its side= x

$\therefore V=x^3$ and $S=6x^2$ (1 mark)

Given $\frac{dv}{dt}=9cm/\sec$ and $x=10$ To find $\frac{ds}{dt}$ (1 mark)

$V=x^3$

Differentiating wrt 't' $\frac{dv}{dt}=3x^2.\frac{dx}{dt}\Rightarrow 9=3(10)^2\frac{dx}{dt}$ $\quad\therefore\frac{dx}{dt}=\frac{9}{3(100)}=\frac{3}{100}$ (1 mark)

$S=6x^2$

Differentiating wrt 't' $\frac{ds}{dt}=12x\frac{dx}{dt}$ (1 mark)

$=12(10)\left(\frac{3}{100}\right)=\frac{18}{5}cm^2/\sec$ (1 mark)

Surface area is increasing at the rate of $\frac{18}{5}cm^2/\sec$

12. A stone is dropped into a quiet lake and waves move in circles at the speed of $5cm/\sec$. At the instant when the radius of the circular wave is 8cm, how fast is the enclosed area is increasing

Answer: Let A be the area and r be the radius of circular wave

$\therefore A=\pi r^2$ at any time 't' (1 mark)

Given $\frac{dr}{dt}=5cm/\sec$

To find $\frac{dA}{dt}$ when $r=8cm$ (1 mark)

Differentiating, $A=\pi r^2$

$\therefore \frac{dA}{dt}=\pi(2r)\frac{dr}{dt}$ (1 mark)

When $r=8$, $\frac{dA}{dt}=2\pi(8)(5)$ (1 mark)

$\frac{dA}{dt}=80\pi\, cm^2/\sec$ (1 mark)

Area is increasing at the rate of $80\pi cm^2/\sec$