Blue Print (As per PU Board)

Topic	1 mark questions	2 marks questions	3 marks questions	5 marks questions	6 marks questions	Total Marks
Integrations	–	2	2	1	1	21

One mark questions

1. **Find the anti-derivative of** $\sin 2x$ **wrt** x

Answer: $\int Sin2x \cdot dx = -\frac{\cos 2x}{2} + C$

2. **Find the anti-derivative of the following w.r.t** x $\frac{1}{\sqrt{9-x^2}}$

Answer: $\int \frac{1}{\sqrt{9-x^2}} dx = \int \frac{1}{\sqrt{3^2-x^2}} dx = sin^{-1}\frac{x}{3} + c$

3. **Evaluate the following** $\int_0^1 \frac{dx}{1+x^2}$

Answer: $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Two marks questions

4. **Evaluate** $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx$

Answer: $I = \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx$

$= \int \frac{e^{\log x^5} - e^{\log x^4}}{e^{\log x^3 - e^{\log x^2}}} dx \qquad = \int \frac{x^5 - x^4}{x^3 - x^2} dx$ (1 mark)

$= \int \frac{x^4(x-1)}{x^2(x-1)} dx = \int x^2 dx = \frac{x^3}{3} + C$ (1 mark)

5. **Evaluate the following** $\int \frac{x-3}{(x-1)^3} e^x dx$

Answer: $I = \int \left(\frac{x-3}{(x-1)^3} \right) e^x dx$

$= \int \left(\frac{x-1-2}{(x-1)^3} \right) e^x dx = \int \left\{ \frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3} \right\} e^x dx$

$= \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx$ (1 mark)

Is of the form $\int e(f(x) + f'(x))dx$ where $f(x) = \frac{1}{(x-1)^2}, f'(x) = \frac{-2}{(x-1)^3}$

$$I = e^x \frac{1}{(x-1)^2} + c$$ (1 mark)

6. **Evaluate** $\int x\sec^2 x dx$

Answer: $I = \int x\sec^2 x dx$

Using parts integration,

$$I = x\int \sec^2 x dx - \int \frac{d}{dx}(x)\left(\int \sec^2 x dx\right)dx + c$$ (1 mark)

$$= x\tan x - \int \tan x dx + c$$

$$I = x\tan x - \log x\, sec\, x + c$$ (1 mark)

Three marks questions

7. $I = \int \frac{dx}{\sqrt{8+8x-x^2}}$

Answer: $i = \int \frac{dx}{\sqrt{8+8x-x^2}}$

$$8+8x-x^2 = -\left[x^2-8x-8\right]$$

$$= -\left[(x)^2 - 2(x)(4) + 4^2 - 4^2 - 8\right]$$

$$= -\left[(x-4)^2 - 24\right] = 24-(x-4)^2$$

$$= \int \frac{dx}{\sqrt{24-(x-4)^2}}$$ (1 mark)

$$= \int \frac{dt}{\sqrt{\left(\sqrt{24}\right)^2 - (t)^2}} \quad \text{put } t = x-4$$ (1 mark)

$$= \sin^{-1}\frac{t}{\sqrt{24}} + c \quad = \sin^{-1}\frac{(x-4)}{\sqrt{24}} + c$$ (1 mark)

8. Evaluate $\int_{-1}^{1} e^x dx$ **using limit of a sum.**

Answer: $\int_{1}^{-1} e^x dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + \ldots + f(a+(n-1)h)\right]$

$$f(x) = e^x, h = \frac{b-a}{n}$$ (1 mark)

$$\int_{-1}^{1} e^x dx = \lim_{h\to 0} h\left[e^{-1} + e^{-1+h} + e^{-1+2h} + \ldots e^{\left[-1+(n-1)h\right]}\right]$$ (1 mark)

$$= \lim_{h\to o} h\left[e^{-1}\left(1+e^4+e^{2h}+\ldots e^{(n-1)h}\right)\right]$$

$$= \lim_{h\to 0} h\left[e^{-1}\frac{e^{nh}-1}{e^{h}-1}\right] \text{ where nh=2}$$

$$\lim_{h\to 0}\left[e^{-1}\left(e^{2}-1\right)\frac{h}{e^{h}-1}\right]$$

$$= e^{-1}\left(e^{2}-1\right)\lim_{h\to 0}\frac{1}{\left(\frac{e^{h-1}}{h}\right)} = \frac{\left(e^{2}-1\right)}{e} = e-\frac{1}{e} \quad \left[\because \lim_{x\to 0}\frac{e^{x}-1}{x}=1\right]$$ (1 mark)

9. Evaluate $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$

Answer: Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \quad ...(1)$

Change $x \to (a-x)$

$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \quad ...(2)$ (1 mark)

(1) + (2) $\Rightarrow 2I = \int_0^a \left[\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} + \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\right]dx$ (1 mark)

$= \int_0^a dx = x|_0^a = a$

$\therefore I = \frac{a}{2}$ (1 mark)

Five marks questions

10. Prove that $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$ **and hence find** $\int \frac{dx}{x^2+2x-8}$

Answer: consider $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)}$

$= \frac{1}{2a}\left[\frac{x+a-(x-a)}{(x-a)(x+a)}\right]$

$= \frac{1}{2a}\left[\frac{x+a}{(x+a)(x-a)} - \frac{x-a}{(x-a)(x+a)}\right]$

$= \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$ (1 mark)

$\therefore \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right]$

$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right]+c$ (1 mark)

$= \frac{1}{2a}\left[\log\left|\frac{x-a}{x+a}\right|\right]+c$ (1 mark)

$\int \frac{dx}{x^2+2x-8} = \int \frac{dx}{x^2+2x+1-9} \quad = \int \frac{dx}{(x+1)^2-(3)^2}$ (1 mark)

$= \frac{1}{2\times 3}\log\left|\frac{t-3}{t+3}\right| + c$ where $t = x+1$

$= \frac{1}{6}\log\left|\frac{x+1-3}{x+1+3}\right| + c = \frac{1}{6}\log\left|\frac{x-2}{x+4}\right| + c$ (1 mark)

11. Prove that $\int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$ hence evaluate $\int\frac{dx}{x^2-6x+13}$

Answer: Put $x = a\tan t$, differentiating, $dx = a\sec^2 t dt$ (1 mark)

$\int\frac{1}{a^2+x^2}dx = \int\frac{1}{a^2+a^2\tan^2 t}.a\sec^2 dt$

$= \int\frac{1}{a^2\left(1+\tan^2 t\right)}a\sec^2 tdt = \frac{1}{a}\int dt$ (1 mark)

$= \frac{1}{a}t + c = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$ (1 mark)

$I = \int\frac{dx}{x^2-6x+13}$, $x^2-6x+13 = (x)^2 - 2(x)(3) + 3^2 - 3^2 + 13$

$= (x-3)^2 + 4$

$\therefore I = \int\frac{dx}{(x-3)^2+4} = \int\frac{dx}{(x-3)^2+2^2}$ (1 mark)

$I = \frac{1}{2}\tan^{-1}\frac{t}{2}+c$ where $t = (x-3)$

$= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c$ (1 mark)

12. Prove that $\int\sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$

Answer: Let $I = \int\sqrt{x^2-a^2}dx$

Using parts integration,

$I = \sqrt{x^2-a^2}\int dx - \int\frac{d}{dx}\sqrt{x^2-a^2}(x)dx + c$ (1 mark)

$= \sqrt{x^2-a^2}(x) - \int\frac{1}{2\sqrt{x^2-a^2}}\cdot 2x(x)dx + c$

$= x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}dx + c_1$ (1 mark)

$= x\sqrt{x^2-a^2} - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx + c_1$

$= x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}dx - a^2\int\frac{1}{\sqrt{x^2-a^2}} + c_1$ (1 mark)

$= x\sqrt{x^2-a^2} - I - a^2\cdot\log\left|x+\sqrt{x^2-a^2}\right| + c_1$ (1 mark)

$\therefore I + I = x\sqrt{x^2-a^2} - a^2\log\left|x+\sqrt{x^2-a^2}\right| + c_1$

$\therefore I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$ where $c = \frac{c_1}{2}$ (1 mark)

Six marks questions

13. Prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. **Hence find** $\int_{-1}^{2}\left|x^3 - x\right|dx$

Answer: Let $\int f(x)dx = f(x)+c$ (1 mark)

LHS $= \int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a) \quad \ldots(1)$ (1 mark)

RHS $= \int_a^c f(x)dx + \int_c^b f(x)dx$

$= F(x)\Big|_a^c + F(x)\Big|_c^b = F(c)-F(a)+F(b)-F(c) = F(b)-F(a) \quad \ldots(2)$ (1 mark)

$\therefore$ from (1) and (2) LHS=RHS

Let $I = \int_{-1}^{2}\left|x^3 - x\right|dx$

$= \int_{-1}^{0}\left|x^3-x\right|dx + \int_{0}^{1}\left|x^3-x\right|dx + \int_{1}^{2}\left|x^3-x\right|dx$ (1 mark)

$I = \int_{-1}^{0} +\left(x^3-x\right)dx + \int_{0}^{1} -\left(x^3-x\right)dx + \int_{1}^{2}\left(x^3-x\right)dx$ (1 mark)

$= +\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Bigg|_{-1}^{0} - \left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Bigg|_{0}^{1} + \left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Bigg|_{1}^{2}$

$= -\left(\frac{1}{4}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+(4-2)-\left(\frac{1}{4}-\frac{1}{2}\right)$

$= -2\left(-\frac{1}{4}\right)+\frac{1}{4}+2 = \frac{1}{2}+\frac{1}{4}+2$

$= \frac{11}{4}$ (1 mark)

14. Prove that $\int_o^a f(x)dx = \int_o^a f(a-x)dx$ **and hence find** $\int_o^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}dx$

Answer: Let $\left.\begin{matrix} a-x=t \\ x=a-t \end{matrix}\right\} \Rightarrow -dx = dt \quad or \quad dx = -dt$

When $x=0, t=a$ and when $x=a, t=0$ (1 mark)

$\therefore LHS = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)dt$ (1 mark)

$= \int_0^a f(a-x)dx$ (1 mark)

$LHS = RHS$

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \quad \ldots(1)$ using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx$$ (1 mark)

$$=\int_0^{\frac{\pi}{2}}\frac{\cos^{\frac{3}{2}}x}{\cos^{\frac{3}{2}}x+\sin^{\frac{3}{2}}x}dx \qquad \ldots(2)$$

$$(1)\ +(2)\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\left[\frac{\sin^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}+\frac{\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}\right]dx$$ (1 mark)

$$2I=\int_0^{\frac{\pi}{2}}dx=x\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}\ \therefore I=\frac{\pi}{4}$$ (1 mark)

15. Evaluate $\int_0^{\pi}\frac{xdx}{a^2\cos^2 x+b^2\sin^2 x}$

Let $I=\int_0^{\pi}\frac{xdx}{a^2\cos^2 x+b^2\sin^2 x} \qquad \cdots(1)$

Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$$I=\int_0^{\pi}\frac{(\pi-x)dx}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$$ (1 mark)

$$=\int_0^{\pi}\frac{\pi dx}{a^2\cos^2 x+b^2\sin^2 x}-\int_o^{\pi}\frac{xdx}{a^2\cos^2 x+b^2\sin^2 x}=\int_0^{\pi}\frac{\pi dx}{a^2\cos^2 x+b^2\sin^2 x}-\mathrm{I}$$ (1 mark)

$$2I=\frac{\pi}{2}\times 2\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$$ (1 mark)

$$2I=\pi\int_0^{\frac{\pi}{4}}\frac{\sec^2 xdx}{a^2+b^2\tan^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\mathrm{cosec}^2 x}{a^2+b^2\cot^2 x}dx$$ (1 mark)

$$I=\pi\left[\int_0^1\frac{dt}{a^2+b^2t^2}-\int_1^0\frac{du}{a^2u^2+b^2}\right]$$

Put $\tan x=t$ in 1st integral, $\cot x=u$ in 2nd integral, $\sec^2 xdx=dt, -\mathrm{cosec}^2 xdx=du$ (1 mark)

$$I=\pi\left[\int_0^1\frac{dt}{a^2+b^2t^2}-\int_1^0\frac{du}{a^2u^2+b^2}\right]=\frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\Big|_0^1-\left(\tan^{-1}\frac{au}{b}\right)_1^0\right]$$

$$=\frac{\pi}{ab}\left[\tan^{-1}\frac{b}{a}+\tan^{-1}\frac{a}{b}\right]=\frac{\pi}{ab}\left(\frac{\pi}{2}\right)=\frac{\pi^2}{2ab}$$ (1 mark)